The Exemplary Career Of Newton's Mathematics

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Newton's remarkable preface to the first edition of the *Principia* presents the fundamental outlook of the new mathematical science of nature: substantial forms have been rejected; natural phenomena will henceforth be studied as manifestations of subsensible particles interacting according to forces that can be experimentally revealed and mathematically expressed, just as the gravitational force, which facilitated calculation of the trajectories of celestial bodies and projectiles.[1] The concepts of particle, force, and calculable trajectory define the edifice of classical physics. This intellectual achievement dominated our view of nature for over two centuries until the quantum revolution revealed that classical physics cannot account for the existence of natural structure.[2] All of this is familiar to students of science today.

But the familiar, taken-for-granted character of what we call Newtonian physics does not comport with what we find if we open Newton's great book, the *Principia*, and try to read it. For we do not find the algebra, calculus, and differential equations in terms of which classical mechanics has long been taught. For example, instead of the standard form of Newton's second law in differential vector calculus, $\mathbf{F} = md^2\mathbf{r}/dt^2$, we find $F \propto QR/SP^2xQT^2$, accompanied by a figure in the style of Apollonius's *Conics*.[3] And from our present perspective, the logical order of Newton's account of "The Motion of Bodies" in Book I of the *Principia*, is elusive. The intention of François De Gandt's book is to help us overcome these difficulties so that we can navigate within the unfamiliar world of Newton's *Principia*. Specifically, De Gandt's

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François De Gandt, *Force and Geometry in Newton's Principia*, trans. Curtis Wilson. Princeton University Press, 1995.

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aim is "to achieve a better understanding of Book I of the *Principia* through a very exact and detailed exegesis of its procedures of reasoning" (61). This statement of purpose is overly modest by its concealment of what is involved. Newton says, concerning forces directed to a central point, that "I here [in the *Principia*] design only to give a mathematical notion of those forces, without considering their physical causes and seats."[4] This sentence identifies the three themes of De Gandt's book: 1) central force; 2) Newton's mathematical treatment of central force; 3) the type of physical-causal neutrality characteristic of Newton's mathematical physics. Let me begin with the latter, which receives the least coverage but is perhaps the most philosophically engaging of De Gandt's themes.

The Causal Neutrality of Newton's Physics

In his preface, De Gandt writes:

> Newton claimed to treat forces in a purely mathematical mode; by deferral, which in a sense turned out to be final, he left in suspense the properly philosophical or physical questions concerning the causes of gravitation and the ontological reality of force. This neutrality (or 'secularism') of 'centripetal' force in face of the controversies on the cause of gravitation is the essential characteristic of the new science. (x-xi)

The term "secularism" here is, needless to say, extremely suggestive for our study of the whole of early modern philosophy in relation to the preceding tradition. On the level of physics, it prompts the question, did Newton then have his own private, extra-mathematical principles of natural philosophy? And, if a crucial part of his physics (the force concept) is *neutral* to, thus compatible with, certain causal accounts, then precisely how is the whole of the Newtonian system not neutral to, but incompatible with, the formal causes of Scholastic natural philosophy, as the preface proclaims? These questions concern philosophy of nature, and thus they fall outside of the more technical focus indicated by the title of De Gandt's book.[5] A brief concluding section, however, returns to the theme of Newtonian secularism.

De Gandt there writes that,

> In the *Principia*, Newton remained very discreet; all his conjectures were entrusted solely to manuscripts, except for the few that appeared in the Queries of the *Opticks*. The theory of the *Principia* is neutral to an astonishing and exceptional degree....was it not the first time that an author writing of natural philosophy concluded a work of this size and significance by confessing that he was unable to find 'the causes' [of universal gravitation]? (270-271)

De Gandt provides an example of the causal neutrality of Newton's force concept in the *Principia*, and then turns to Query 28 of the *Opticks*, a *locus classicus* for Newton's conjectures. The example is the following: in the Scholium to Prop. 69, Newton lists the ways in which bodies could attract each other according to the law of gravitational force: by "the action of bodies, either in seeking to approach each other or in agitating each other by emitted spirits; or...[by] the action of...any medium whatever, whether corporeal or incorporeal." In the *Principia*, Newton does not decide among these alternatives; he does not need to in order to carry out his mathematical treatment of forces and motions. In Query 28 of the *Opticks*, Newton rules out a "dense Fluid" medium on grounds that it would retard the motions of planets and comets, an effect that we do not observe. De Gandt then quotes from the last paragraph of Query 28, wherein Newton rejects "Hypotheses for explaining all things mechanically." We ask, what does Newton mean by the term "mechanical," and what non-mechanical hypotheses or principles might he thus entertain?

Unfortunately, a well-known caution is here in order. It is not easy to discover Newton's understanding of first causes and moral philosophy, and the relation of that understanding to his natural science. Of his published writings, Queries 28, 30, and 31 of the

Opticks, and the General Scholium of the *Principia* contain basic texts on this issue. The difficulty for Newton seems to be that, in the now familiar interplay between theory and experiment, we can find increasingly elementary particles and increasingly general laws of nature, but this analysis does not, of and by itself, give rise to a hierarchical ascent to first causes that have any relation to human moral phenomena.[6] Thus, against an extensive historical background that demanded a public account of the positive relation between ultimate principles and human good, Newton would be under pressure to defend the moral significance of his immensely powerful physics, hence to assure a growing audience that the secularism of his science is not his last word. And so we have his declarations concerning "the dominion of God" (General Scholium) and the value of natural philosophy in knowing "what Benefits we receive from [God]...our Duty towards him [and] towards one another" (Query 31, last paragraph).[7] It is not clear, however, exactly how these assertions follow from the demonstrative content of the *Principia*. And Newton's emphasis on comets in Query 28 and in the General Scholium is puzzling: In the final paragraph of Query 28, he juxtaposes the question, "Whence is it that Nature doth nothing in vain; and whence arises all that Order and Beauty which we see in the World?" with the question, "To what end are Comets, and whence is it that Planets move all one and the same way in Orbs concentrick, while Comets move all manner of ways in Orbs very excentrick...?" These different visible patterns of motion (planetary and cometary) fall under the same Newtonian mathematical description, and thus exemplify the economy of nature posited in Rule 1.[8] But do comets show that nature acts for an end; aren't they a problem (along with all the other planets in the solar system besides earth) for this notorious Aristotelian claim (*Phys*. II.8)? Newton indeed continues in Query 28 with the classic (and abiding) examples of natural purpose, the structure and function of biological systems: "Was the Eye contrived without Skill in Opticks, and the Ear without Knowledge of Sounds?" But he then concludes with his analogy between the

"Sensory of Animals" or "our little Sensoriums," on the one hand, and infinite space as the sensorium of God, on the other. Concerning this unusual analogy, he asks, "does it [the analogy] not appear from Phaenomena...?"—surely not as persuasively as the self-evidently purposive designs of an eye or an ear. Finally, in the General Scholium (published seven years after the material in Queries 25-31), the biological examples disappear in favor of two references to "final causes," and the final paragraph on the "most subtle spirit" whereby "all sensation is excited, and the members of animal bodies move at the command of the will, namely, by the vibrations of this spirit, mutually propagated along the solid filaments of the nerves, from the outward organs of sense to the brain, and from the brain into the muscles." This most subtle spirit, with which the *Principia* concludes, is a Newtonian hypothesis that accords with the particles-and-forces model with which the *Principia* begins (first preface). Apparently it is a *non-mechanical* hypothesis—thus not subject to his strictures against mechanical hypotheses—because the particles of this subtle medium, or ether, are exceedingly *active*. That is, if we take the term "mechanical" to mean Cartesian corpuscles *passively* colliding among themselves, then we can reconcile Newton's rejection of (Cartesian) hypotheses with Newton's use of (Baconian) hypotheses. But what, then, is the relation between active principles and first causes? Isn't the most subtle spirit, which is intended to explain animate motions (among other things), just as deterministic as any forces-and-particles account? Newton's active ether may satisfy Leibniz's demand that we find "intelligible means" whereby God produces natural effects, but now we have an acute problem of the relation between rational will and bodily motion in the human animal. May we suppose that Newton believed in the preestablished harmony or some other notion of psycho-physical parallelism?

It is surely no criticism of De Gandt that he does not pursue this inquiry; it would constitute another book. He affirms (270) that "[t]here would be a place in [Newton's] universe for various 'active principles'."[9] He then concludes *Force and Geometry* with a final, two-page comment on the neutrality or secularism of Newton's force concept. He says:

> Through Newton's work, a certain level of theory became autonomous. Once the mathematical physicists have determined the center of forces, once they know the law of the variation of the force [e.g., inversely as the square of the distance from the force center] and have succeeded in unifying certain families of phenomena [e.g. celestial and terrestrial local motions], they can, on their own accounts [privately], reflect on 'causes' or 'physical reasons' (to use Newton's term); but that will have no effect on the unfolding of their reasonings in mechanics or dynamics. (271)

Accordingly, "the very notion of force had a different meaning" for Euler, for Boscovich, and for D'Alembert, yet all contributed to the development and systematization of Newtonian physics (271). Can we identify a seminal or paradigmatic example of the causal neutralization of the concept of central force in Newton's own writings? Here we are well served by De Gandt's expertise: he isolates Newton's treatment of the law of areas, Prop. I of the *Principia,*[10] as "the emblem of a new conception of force, disembarrassed of phantasms and indifferent to physical causes," seen by Newton as "a theorem of the highest importance, striking and pregnant with consequences" (272). Given the appropriate mathematical analysis of trajectories, one can determine the presence of a central force effecting the motion by showing the uniform sweeping out of areas around a certain point in space: such a point must be a center of force. But *what is* a center of force? "It is not necessary to know the nature of this central point or by what it is occupied [!] nor to construct hypotheses as to the action exercised by a source or an attracting body" (272). Once we are able to take a central force as a mathematical entity, we can forget what a strange type of intelligible it is.[11] Let us then turn to De Gandt's expertise in the history of exact science, to his richly detailed work on the mathematization of motion and force.

The Geometrization of Central Force: *De Motu*

De Gandt's stated purpose is to enable us "to achieve a better understanding of Book I of the *Principia*" (61). This is accomplished very successfully through his translation with commentary of Newton's 1684 *De Motu.* This short writing was prompted by Halley's question to Newton, what curve would the planets describe if they were attracted to the sun by a $1/r^2$ force? (7) It is of course known from Kepler that the planetary orbits are elliptical; but could one derive this in a mathematical way from the candidate $1/r^2$ attractive force? Newton's answer to Halley, essentially the *De Motu,* consists of three definitions, four hypotheses, four theorems, and seven problems, which form the kernel that Newton develops by 1686 into Sections I-III and VII of Book I of the *Principia.*[12] As such, the *De Motu* provides an "initial orientation...a pinpointing of principal results, connections, and methods" (14) indispensable for the reading of the *Principia.* The three most important items in the *De Motu* are: the generalization of Galileo's law of fall (Hypothesis 4; *Principia,* Lemma 10); the law of areas, mentioned above (Theorem 1; *Principia,* Prop. 1); the resulting expression for the instantaneous force on a moving body in terms of the local curvature of its trajectory (Theorem 3; *Principia,* Prop. 6, Cor. 1). Newton expresses this latter relation in his distinctive, kinematic-infinitesimal geometry as $F \propto QR/SP^2 x QT^2$, where SP is the radius vector from the force center to the position of the body at time t, and Q is its position at time $t+\Delta t$. Unlike an equation in algebra or calculus, Theorem 3 is unintelligible without looking at the associated figure (*Force,* 31; *Principia,* 48). Theorem 3 holds as we take "the ultimate quantity...when the points P and Q come to coincide" (31). As De Gandt rightly puts it, Newton's "is a very special geometry whose nature must be described" (x). I comment on these three items, and then discuss De Gandt's treatment of the emblematic law of areas.

Hypothesis 4 is the generalization of Galileo's law of fall from constant forces (producing uniform acceleration) to spatially varying forces. It is based on tracing the Galilean law—distance traversed by a body is proportional to time squared—to "the very beginning of its motion" (18). The requirement for proof of Hypothesis 4 (not supplied in *De Motu*) is an important source of the differential calculus, pioneered by Newton (without using differential notation) in Lemmas 6-9 of the *Principia,* on ultimate ratios of chords, arcs and tangents.[13] It is important to see the logical connection between Hypothesis 4, and Theorems 1 and 3 of *De Motu,* and De Gandt explains this in Newton's own terms on pp. 31-32 of *Force.* Schematically stated in more contemporary notation, Hypothesis 4 reads, $F \propto QR/dt^2$, where the line QR is the body's deviation in time increment dt from rectilinear inertial displacement along the tangent to the orbit, and F is the magnitude of the force responsible for this incurving motion.[14] De Gandt describes QR as the "very small—nascent or infinitesimal—trajectory of fall; it is the line that the body would traverse during the [infinitesimal] time considered under the sole action of the centripetal force if it started from rest" (32). "QR is therefore the index and geometrical measure of the force that draws P towards S" (11). Finally, the law of areas, Theorem 1, is $dt \propto dA$, for any central force, where dA is the increment of area swept out in time increment dt by the radius vector from the force center to the moving body. Theorem 3 follows by substituting dA in place of dt in the generalized (infinitesimal) law of fall: thus $F \propto QR/dA^2$. The significance of this is that the expression for the force is now completely *geometrized* :

> The *De Motu*...contains neither the three famous laws of motion nor any mention of absolute time and space but presents, above all, the first evidence of a geometrical translation of central force. This is its primary interest: how did force come to be expressed

> geometrically, and how did mathematics become capable of translating dynamics? (14-15)

Theorem 3 of *De Motu* is the seed of a major part of the *Principia*, for it is Newton's expression for what becomes his second law of motion, as that law applies to central forces—the main subject of Books I and III of the *Principia* (wherein velocity-dependent forces, e.g. air resistance, are neglected). It is seminal for the following two reasons. First, Theorem 3, in combination with knowledge of the shape of a moving body's orbit, permits the derivation of the force law (the force as a function of distance) required to produce the given type of orbit. Illustrations of this procedure—for solution of the so-called "direct problem"—are given in Problems 1, 2, and 3 of *De Motu*, which become Props. 7 (Cor. 1), 10, and 11 of the *Principia*. Second, Theorem 3 (along with additional results) in combination with knowledge of the force law allows calculation of the orbit or trajectory resulting from given initial conditions. This procedure—for solution of the so-called "inverse problem"—is illustrated in Problems 4 and 5 of *De Motu*, which become Props. 17 and 32 of the *Principia*. The force law is restricted, however, to $1/r^2$.

This completes my sketch of the logical connections between the most important items of *De Motu* and their ramifications in the *Principia*. But I have oversimplified a major issue in Newton and post-Newtonian physics, an issue that De Gandt features in the latter part of *Force and Geometry* (244-264). It is this: The quest for solution of the general inverse problem (from *any* central force, F(r), derive resulting trajectories), in fact, carries Newton and post-Newtonian physics beyond the distinctive, kinematic-infinitesimal geometry of Sections I-VII of Book I of the *Principia*. For, in the face of the general inverse problem, that geometry "become[s] too narrow a framework for the study of forces" (264). De Gandt thus chronicles not only the rise of Newton's geometry (from the *De Motu* to the *Principia*) but also its decline (within and after the *Principia*). Let us look at De Gandt's treatment of the law of areas in *De Motu*, and then turn to the end of the story, the supersession of Newton's distinctive mathematics by the now familiar analytico-algebraic calculus.

The law of areas for any central force is: "Theorem 1. All orbiting bodies describe, by radii drawn to the center, areas proportional to the times" (22). Of course "Kepler had formulated this law in his *Astronomia nova* of 1609...[thus] Newton was redemonstrating a well-known principle. But to put the matter in its true light, Newton's demonstration was incredibly new" (23). De Gandt gives Newton's proof in twenty lines of text and a figure, with commentary following. (The proof is essentially the same as that in *Principia*, Prop. 1, with two exceptions: in the *Principia*, Newton makes explicit that motion under a central force occurs in a fixed plane, and he appeals to Lemma 3, Corollary 4, about which more below.) Newton's proof is both elegantly simple and remarkable; he demonstrates "with an unprecedented economy of means—at least if one accepts the passage to the limit, by which the demonstration is concluded" (23). I focus on the latter.

Newton begins his proof not with a continuously acting central force, but rather with a succession of discrete, instantaneous impulses occurring at equal time intervals and directed to one point, S. "The orbit is [consequently] made up of a series of rectilinear segments, and at each vertex of the polygonal path the body is 'deflected' by an impulsion" (25). Given this premise, the demonstration becomes that of proving the equality of all the contiguous triangles defined by inertial motion between two successive impulses and the two radii from S to those points of impulse. (The figure is given in *Force*, 22, and *Principia*, 40.) The proof follows easily from two applications of *Elements*, I.38, the equality of triangles with the same heights and bases.[15] So far, Newton's argument is elegantly simple. His conclusion, the passage to the limiting "real case" (25) is remarkable: "Now let these triangles be infinite in number and infinitely small [thin], so that each triangle corresponds to a single moment of time, and with the centripetal force then acting unremittingly, the proposition will be established" (22). De Gandt: "How is the demonstration to be extended to cover the less simple case of a curvilinear orbit with a centripetal force acting continuously? Newton did not show himself very hard to please on this point: it sufficed to take very small intervals of time!" (26) Aside from older questions about actual infinities, there is an obvious objection to Newton's concluding move: every triangle, no matter how thin, has a base defined by two endpoints, which must here correspond to two distinct force impulses. So, in strict precision, the force never acts continuously. The corresponding language of the *Principia* is somewhat improved: "let the number of those triangles be augmented, and their breadth diminished *in infinitum* ; and (by Cor. 4, Lem. 3) their ultimate perimeter...will be a curved line: and therefore the centripetal force...will act continually (*indesinenter).*" Here we have no claim for an actual infinity of triangles but rather for an infinite sequence (*in infinitum*) of the sort that becomes constitutive of modern analysis. Let us briefly consider, then, Lemma 3 of the *Principia*, on ultimate sums of parallelograms, and the ultimate ratios of those sums to the curvilinear figure to which they converge. My intention is to make clear that a new sense of precision (relative to ancient science) is at the foundation of Newton's physics, a sense intimately related to practice.

In Lemma 3 (or in any of Lemmas 2-4) we find figures now familiar in texts on integral calculus. A number, n (a positive integer), of parallelograms are inscribed and circumscribed in order to bound a curvilinear figure. As n increases without limit, we have two sequences of sums that converge to the area of the curvilinear figure from above and below such that for "any given difference," D, in the language of Lemma 1, we can find an n for which the differences between the sums and the area of the curvilinear figure are less than D; the sums therefore "become ultimately equal" to the figure and to each other. (The sums converge to the area under the curve, and their perimeters converge to the curve itself; the latter is Corollary 4 of Lemma 3.) The sequences of parallelogram sums are asymptotic (in area and shape) to the given figure, and so, as Leibniz says, "there can be no ultimate difference, nevertheless they never become equal."[16] Indeed, it is easy to show (at least for certain figures) that if, for any finite n, we set the sums precisely equal, we get 1=0.[17] Clearly the key to avoiding this contradiction lies in the meaning of the crucial term "ultimately" in Newton's pregnant phrase "ultimately equal" in the statement of Lemma 1. I suspect, however, that whatever we make of "ultimately" (through the theory of limits) we still have two distinct, true statements: first, the sums and their limiting figure are never precisely equal for any finite n, and if we deny this we get 1=0; second, despite the truth of the first statement, the difference between the sums and the figure can indeed be made smaller without limit. Which of these two statements is more important? I believe the answer depends on one's purpose. For ancient science (Plato, Aristotle, Euclid), is not the first statement more important because the purpose is to know, to the extent we are able, *what is* eternally? For modern science (as it emerges in the 16th and 17th centuries), is not the second statement more important because the purpose is to know *what we can make* in time? Who gives "that [very small] given difference"?[18] Is it not the engineers, who specify the error window, acceptable tolerance, or accuracy required on a controlled physical process in function of the practical objectives? For example, we will not have to place a future probe at a *mathematical point* on the surface of Mars but only within a small *but extended* region of diameter D about a point. More generally, measurements of *physical*, as opposed to *mathematical*, magnitudes are always of limited numerical precision, accurate to ±D. (Does this not arise from the nature of matter itself, from the distinction between the sensible and the intelligible—a distinction crucial for ancient philosophy and obliterated by Descartes?) Hence, calculational techniques yielding differences or errors less than D are good enough, relative to the practical problem at hand. Despite its wonderfully mathematical character, is not Newton's physics informed more by *praxis* and *technê* than by *theôria* ? This deep-seated orientation to practice would seem to go hand in hand with

Newton's causal neutrality or secularism: precision, both in the sense of Euclidean *precise equality*, and in the sense of *Republic* 504e1-3, is replaced by real-numerical precision, i.e. a sufficient number of significant digits in calculation and measurement.[19] Perhaps this is why Newton does not "show himself very hard to please." As alluded to above, and discussed below, De Gandt will enable us to see how the primacy of problem-solving technique affects the career of Newton's mathematics within the *Principia* itself.

Following his presentation of Theorem 1, the law of areas, De Gandt takes up the remaining theorems and problems of the *De Motu* (26-54), and then summarizes five characteristics of Newton's distinctive geometry (55-56). I paraphrase the first three of these as follows:

> 1.The evaluation of the distance traversed is valid only at the beginning of motion, only for the nascent displacement (Hyp. 4). Hence the measure of the force in terms of QR/SP^2xQT^2 holds only if the figure QRPT is indefinitely small, only as P and Q coincide (Th. 3).
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> 2.What holds for a polygonal surface composed of a finite number of triangles can be extended to an infinite number of infinitely small triangles that compose a curvilinear surface (Th. 1).
>
> 3. In ratios and proportions it is permitted to substitute some lines for others from which they differ very little as two points approach and coincide (Th. 2, Probs. 1, 2, 3).

In all of these procedures certain "relations [are] preserved while the figures are deformed to an ultimate configuration....all presuppose motion and time" (56).[20] Concerning the third of these, De Gandt says:

> A simple extension of classical geometry would suffice for the third case: certain relations are enunciated with respect to finite, immobile figures, and the configura-

> tions are then deformed in such a way as to make possible the replacement of one line by another line from which it differs very little. Proportions and relations are thus enriched by authorizing the substitution of one magnitude for another if the two are infinitely close to equality. (56)

Hence the kinematic-infinitesimal character of Newton's ingenious geometry. But, in keeping with my remarks in the preceding paragraph, can we not question whether classical geometry would agree that such an authorization yields enrichment simply; is there not a certain trade-off? In terms of the being of magnitude as divisible (*Meta.* 1020a7-10), and the strict meaning of equality (1021a7, 12), it is the case that either the two magnitudes are unequal, or, when the two points are coincident and thus no longer two, the two magnitudes do not exist.[21] Hence, as before (on Lemmas 2-4), we must say that they are never actually equal, but possess "the ultimate ratio...of equality" (Lemma 7). It seems, again, that an adequate account of this issue would have to consider (among many other things) whether there is a difference, as Aristotle holds, between mathematical and physical magnitude in terms of motion and time. Is not the "physicalization" of geometrical magnitude (its becoming mobile and temporal) conspicuous in Newton, as the essential concomitant to the geometrization of physical magnitude? Let us turn to the end of the story, to De Gandt on "the emergence of a new style" (244) in Newton's *Principia.*

Newton's Geometry Superseded: the General Inverse Problem

As previously stated, the general inverse problem is: given any central force law, and the initial position and velocity, r_0, v_0, of a body relative to the force center, determine the subsequent trajectory of the body. The first beginnings of Newton's solution of the inverse problem, restricted to a $1/r^2$ force, are illustrated in Problems 4 and 5 of *De Motu,* which become Props. 17 and 32 of the *Principia.* Newton's turn to the general inverse problem begins only later, in Props. 39 and 40 of the *Principia,* with a "new approach [that] corresponds to what will later be called conservation of energy" (246). De Gandt indeed tracks the transition within the *Principia* from Newton's distinctive geometry, which requires "an active intervention, an exercise in 'seeing'," (244) to "a more abstract and less geometrical way [of treating force] that is closer to the analytico-algebraic style that will flourish in the eighteenth century" (246), closer, that is, to "the more or less automatic methods of the 'infinitesimal calculus' " (244). Prior to Props. 39 and 40, "it [was] necessary to know the trajectory in order to study the force, since the latter was manifested and measured by the divergence between the tangent and the actual trajectory" (247). Hence, the geometric approach based on Theorem 3 of *De Motu* and Prop. 6, Cor. 1, of the *Principia,* in fact, suffers from a defect. Another approach is needed for the more general case in which the force law is given but nothing is known of the trajectory (other than initial position and velocity). With Props. 39 and 40, "it becomes possible to study the effects of force without knowing the moving body's trajectory" (247). In Props. 40 and 41, Newton begins the calculation of the trajectory from the force law. In these propositions, force is no longer accessible through an image that resembles its temporal effect on motion (the figure QRPT), but rather represented as a Cartesian curve relative to coordinate axes, i.e., a function of distance. Newton's mathematization of force has shifted ground.[22] According to De Gandt, the demonstration of Prop. 39 comes close to the methods of calculus; he singles out the passage in which Newton specifies the central force essentially as $F \propto dv/dt$: "the force will be directly as the increment...of the velocity and inversely as the time." For De Gandt, "[t]his passage of Book I [Prop. 39] is perhaps the one in which the emergence of a differential and analytic style is most evident" (253). De Gandt then discusses Newton's demonstrations of Props. 40 and 41, and their development at the hands of Varignon and John Bernoulli (253-264). Due to the superior generality of his analytical technique, it is John Bernoulli, not Isaac Newton, who best solves the general inverse problem (264).

Is not the fate of mathematical physics after Prop. 40 of Newton's *Principia* a clear example of "the intimate connection between the mode of 'generalization' of the 'new' science and its character as an 'art' " that stands near the center of Jacob Klein's reflections on the history of mathematics?[23] In spite of its imprecisions and heterodoxy compared to classical mathematics, Newton's distinctive geometry permits a certain intuitive clarity in a demonstration; it enables us to *see the reason* for the conclusion in the analysis of motion. It is as if Newton at first tried to forestall the modern movement toward mental machine technique but then gave up—by giving in to the need for more general mental tools. After all, did he not write in the First Preface of the *Principia* that "geometry is founded in mechanical practice" and is a "part of universal mechanics"? On this account, the mechanical algorithm for calculation would seem to be just as good as geometry for the description, prediction, and control of physical process. As David Lachterman says, "[t]he mechanization of nature advances *pari passu* with the machinations of the mind."[24]

Force and Geometry in Newton's Principia is an important book for the light it sheds on the rise and fall of Newton's geometry. But it offers more than this, and considerably more than I have discussed here. De Gandt is an extremely competent specialist in the history of exact science from the 17th through the 18th centuries—a kaleidoscopic period. In the second chapter, which follows his rendition of Newton's *De Motu,* De Gandt gives detailed accounts of the varied concepts of force in Kepler, Galileo, Descartes, and Huygens, along with short but valuable discussions of Barrow,[25] Torricelli, Hooke, and Flamsteed (58-158). The dominant figure is, not surprisingly, Galileo. The third and final chapter is titled "The Mathematical Methods" (159-264). Here De Gandt gives us a rare descriptive catalogue of the bewildering variety of new mathematical techniques that arise in and after Galileo. These are the methods of indivisibles, of ultimate ratios and finite witnesses, of fluxions, and finally of the infinitesimal calculus. The problem of the composition of the continuum forms a common theme, and indeed one could write a substantive review of De Gandt's book from this perspective alone.

Finally, it is a pleasure to report that Curtis Wilson's translation is excellent. The content is extremely clear and leaves no sense that one is reading a foreign author.

Notes

1. Isaac Newton, *Sir Isaac Newton's Mathematical Principles of Natural Philosophy,* trans. Andrew Motte and Florian Cajori (New York: Greenwood Press, 1969), xvii and xviii. (My citations from the *Principia* will be taken from this edition, hereafter referred to as *Principia.)* See also the first four paragraphs of Roger Cotes's preface to the second (1713) edition, *Principia*, xx-xxi.

2. By mid- 19th century, faith in the classical mechanical view was unquestioned by scientists of the stature of Helmholtz: "Finally, then, the task of the physical natural sciences is...to reduce natural phenomena to unchanging attractive and repulsive forces, whose strength depends only on the distance [between particles]. The realizability of this task is, at the same time, the condition of the complete comprehensibility of nature." "Über die Erhaltung der Kraft [1847]," *Wissenschaftliche Abhandlungen* (Leipzig, 1882), Vol. I, 15-16. Contrast this with Bohr's statement to Heisenberg in 1922: "[T]he stability of matter [is] a pure miracle when considered from the standpoint of classical physics." Heisenberg, *Physics and Beyond,* trans. Arnold J. Pomerans (New York: Harper Torchbooks, 1972), 39. It is a good question whether the mechanical view would have become so sure of its own comprehensive adequacy if it had focused more on biological phenomena, on the kinds of beings to which Aristotle mainly looked in his account of nature (e.g., *Meta* 1032a19).

3. *Principia*, Prop. VI, Cor. I, 48. In this review, boldface letters represent vectors.

4. *Principia*, Def. VI11, 5.

5. Concerning the question of how form is ruled out of Newton's physics, I believe it can be shown that the parallelogram rule for composition of forces, Corollary II of the *Principia,* plays the pivotal role in defining a reductionist

whole-part relation that is strictly incompatible with any holistic principle, and thus with formal causality in nature.

6. In the forces-and-particles model of the universe proposed in the first preface of the *Principia* as a program for future research into natural phenomena, it is hard to see what the specifically human consists in. Were this model adequate to nature, a human being as such would cease to be a natural being, because nature would lack all specifically human content.

7. *Principia*, 544; *Opticks*, 405. In 1710 Berkeley criticised Newton's doctrine of absolute space, time, and motion as theologically "pernicious"; see *Principia* 668, Cajori's note 52. The General Scholium was added to the second (1713) edition of the *Principia*. The content of Queries 25-31 of the *Opticks* appeared in the Latin edition of 1706 and in the subsequent English editions of 1717, 1721, and 1730; see I. Bernard Cohen's preface to *Opticks*, xxxi.

8. *Principia,* 398. Thus nature does nothing in vain in the sense that, in the myriad effects it produces, the smallest number of causes are employed; this is what mathematical physics discovers. It does not follow, however, that these effects can be recognized by us as *ends* or particular goods in the visible universe (they might be very bad, like a killer asteroid). Perhaps, for Newton, the intelligible economy that we can discover in nature's use of causes is the sign of an intelligence that aims at a universal good, which we may be unable wholly to discover, but in which we can plausibly believe.

9. De Gandt cites (270, note 3) J. E. McGuire,"Force, Active Principles and Newton's Invisible Realm," *Ambix* 15 (1968), 154-208. Some older sources may also be relevant: on Newton's interest in Jacob Boehme, see David Brewster, *Memoirs of Sir Isaac Newton* (Edinburgh, 1860), Vol. 2, 371, and for the influence of Henry More's Neoplatonism on Newton, see A. J. Snow, *Matter and Gravity in Newton's Physical Philosophy* (London: Oxford Univ. Press, 1926).

10. We originally learn the law of areas in the form of Kepler's second law: the radius vector from the sun to each planet sweeps out equal areas along the elliptical orbit in equal times. But conic-sectional orbits correspond to an inverse-square force directed to a focus. As we shall see, Newton's demonstration of the law of areas is superior because it covers the general case of *any* central force (one directed to a point and whose strength depends only on the distance to that point), not just an inverse-square force. In standard presentations, a central force is a sufficient condition for conservation of orbital angular momentum, from which the law of areas is then derived.

11. It is strange because it represents a principle of motion in nature that is indifferent to the kind, size, shape, internal structure and function of the two interacting bodies. Normally, the way we expect a body to move is intimately related to its kind or species. Newton's gravitational law is radically species- or structure-neutral. If that law was the paradigm for the fundamental principles of nature, i.e. for what gives rise (via subsensible particles) to the visible species of bodies, then perhaps we can begin to understand Newton's astonishing assertion that, "Every body can be transformed into another, of whatever kind, and all the intermediary degrees of qualities can be successively induced in it." *(Principia,* 1st edition, Hypothesis III.) See Alexandre Koyré and I. B. Cohen, eds., *Isaac Newton's Philosophiae Naturalis Principia Mathematica* (Cambridge, MA: Harvard Univ. Press, 1972), 550.

12. There are several versions of the *De Motu.* De Gandt focuses on *De motu corporum in gyrum,* given in MS B of A. R. and M. B. Hall, eds., *Unpublished Scientific Papers of Isaac Newton* (Cambridge: Cambridge University Press, 1962), which, according to De Gandt, is probably the earliest (*Force,* 15). Problems 6 and 7 of the *De Motu* concern motion in resistive media, the subject of Book II of the *Principia,* and are not discussed by De Gandt. The correspondence between the hypotheses, theorems and problems of *De Motu* and the lemmas and propositions of the *Principia* is given by De Gandt at *Force,* 275-276.

13. For these lemmas, and thus for Newton's kinematic-infinitesimal geometry, it is helpful to read the section titled "Comments on the Illustrations," in Thomas K. Simpson, "Newton and the Liberal Arts," *The College* (January, 1976), 7- 10.

14. In the language of post-Newtonian vector analysis, $QR = |\mathbf{r}(t) + \mathbf{v}(t)\Delta t - \mathbf{r}(t+\Delta t)|$ as $\Delta t \rightarrow dt$, where $\mathbf{v}(t) = d\mathbf{r}/dt$, the velocity. Expanding $\mathbf{r}(t+\Delta t)$ to second order in Δt, we obtain $QR = |\mathbf{a}(t)|dt^2/2$, where $\mathbf{a}(t) = d^2\mathbf{r}/dt^2$, the acceleration. This is the differential form Galileo's law of fall, $x = at^2/2$. See also S. Chandrasekhar, *Newton's Principia for the Common Reader* (Oxford: Clarendon Press, 1995), Sec. 20, and J. Bruce Brackenridge, *The Key to Newton's Dynamics* (Berkeley: Univ. of California Press, 1995), 211-216.

15. The equality of time intervals between impulses is necessary for the equality of bases in the first application of *Elements* I.38, and the centripetal direction of each impulse is necessary for the equality of heights in the second.

16. Leibniz, *Marginalia in Newtoni Principia Mathematica,* ed. E. A. Fellmann (Paris: J. Vrin, 1973), 27.

17. For example, define the figure by the line $y = 1-x$ on the unit interval. The area is $A = 1/2$. Let the approximating parallelograms have bases 1/2, 1/3,...1/n. Then, for any n, the area of the associated sum of circumscribed parallelograms is $(1+ 1/n)/2$, and of the inscribed parallelograms $(1 - 1/n)/2$. Clearly, the limit as n goes to infinity of each is $A = 1/2$. Yet equating $(1 \pm 1/n)/2$ to 1/2 yields $1/n = 0$. Multiplying by n (which is finite) we have $1 = 0$.

18. *Principia*, Lemma 1, 29.

19. See Jacob Klein, "On Precision," in Robert B. Williamson and Elliott Zuckerman, eds., *Jacob Klein: Lectures and Essays* (Annapolis: St. John's College Press, 1985), 289-308. When Newton says that, "the errors [in mechanical arts as opposed to geometry] are not in the art, but in the artificers" *(Principia,* First Preface, xvii), he seems to be taking the Cartesian rationalist position, to be canceling the ontological distinction between physical magnitudes (e.g., bodies) and geometrical magnitudes. But I believe he is saying something rather different: there may or may not be a distinction between the physical (sensible) and the geometrical (intelligible), but no such distinction could ever matter for our practice of measurement and calculation. Recent results in nonlinear dynamics (so-called chaos theory) call this notion into question; see Joseph Ford, "How Random is a Coin Toss?" *Physics Today* 36 (April 1983), 40-47.

20. See also *Force*, 161-167, and Simpson, "Newton and the Liberal Arts," 9.

21. If we say they exist and are actually equal, can we not infer a contradiction, like 1 =0? Note that this is not quite the same as the two objections to which Newton responds in his Scholium after Lemma 11 of the *Principia,* namely, that either there are no ultimate ratios, or, if there are, there must be ultimate magnitudes *(Principia,* 38-39).

22. Concerning the methods of calculus, what could be more automatic than the algebraic manipulation of differentials currently employed in the derivation of the work-energy theorem (the basis of conservation of mechanical energy)? $F=mdv/dt$; therefore, $Fdx=m(dv/dt)dx=m(dx/dt)dv=mvdv=md(v^2/2)$; integrating, therefore, $\int Fdx=mv_f^2/2-mv_i^2/2$, where v_f and v_i denote final and initial velocities. Look at the switching of the differentials, dv and dx, in the third step.

23. Jacob Klein, *Greek Mathematical Thought and the Origin of Algebra,* trans. Eva Brann (Cambridge, MA: MIT Press, 1968), 122.

24. David R. Lachterman, *The Ethics of Geometry* (New York: Routledge, 1989), 71. Klein and Lachterman give us a nexus of great philosophical issues

centered on the transition from mathematics as contemplative to mathematics as operative. But the differences between them appear to be significant. Klein's major theme is the symbolic number concept, for which the comparison of Diophantus and Vietá (against a Platonic backdrop) is the textual linchpin; Lachterman's major theme is geometric construction, for which the comparison of Euclid and Descartes is most crucial. Klein's description of symbolic concept-formation in the transition from Apollonius to Descartes, in "The World of Physics and the 'Natural' World," *Lectures and Essays,* 1-34, impinges on Lachterman's theme but is, apparently, not the same, because, for Lachterman, symbolic abstraction is "only a preparatory first step" *(Ethics,* 186). The project of understanding and relating their differing accounts is permanently fascinating and discouragingly complex.

25. De Gandt gives a long quotation from Barrow *(Force,* 109-111) that is striking for the fastidiousness with which Barrow attempts to secure concepts of distance, time, velocity, and force as *quantified* regardless of whether they are ultimately discrete or continuous. Clearly the "secularization" of physics was basic to the context in which Newton worked.